Gravitational wave constraints on post-inflationary phases stiffer than radiation

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We point out that the existence of post-inflationary phases stiffer than radiation leads to the production of stochastic gravitational wave (GW) backgrounds whose logarithmic energy spectra (in critical units) are typically ''blue'' at high frequencies. The maximal spectral slope (for present frequencies larger than 10^{-16} Hz) is of order 1 and is related to the maximal sound velocity of the stiff plasma governing the evolution of the geometry. The duration of the stiff phase is crucially determined by the back reaction of the GW leaving the horizon during the de Sitter phase and reentering during the stiff phase. Therefore, the maximal (inflationary) curvature scale has to be fine-tuned to a value smaller than the limits set by the large scale measurements ($H_{\rm dS} \lesssim 10^{-6}\ M_P$) in order to have a sufficiently long stiff phase reaching an energy scale of the order of 1 TeV and even lower if we want the stiff phase to touch the hadronic era (corresponding to $T_{\rm had} \sim 140$ MeV). By looking more positively at our exercise we see that, if an inflationary phase is followed by a stiff phase, there exists the appealing possibility of ''graviton reheating'' whose effective temperature can be generally quite low. [S0556-2821(98)02318-2]

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I. INTRODUCTION

Strong causality arguments [1] forbid the existence of a never ending radiation dominated epoch. If this were the case regions emitting a highly homogenous and isotropic cosmic microwave background radiation (CMBR) at the decoupling epoch would not have been in causal contact in the far past. This problem of horizons (together with other kinematical problems of the standard cosmological model) led us to assume an inflationary phase [2] of accelerated expansion ($\dot{a}>0$, $\ddot{a}>0$) where the (effective) equation of state describing the background sources during inflation was drastically deviating from the one of radiation.

In the context of ordinary inflationary models [2] the transition from the inflationary regime to the radiation era is usually associated with a reheating phase where the energy density of the inflaton field is released ''producing'' a radiation dominated phase. The dynamics of the inflaton right after the inflationary epoch has been recently discussed in detail [3].

If this is the dynamical picture of the evolution of our Universe in its early stages, then one of the most interesting (and most difficult to test) implications is the production of a stochastic background of gravitational waves (GW). It has indeed been known for many years that the various transitions of the curvature scale lead, necessarily, to the amplifications of the quantum-mechanical (vacuum) fluctuations of the tensor modes of the geometry [4] and to the consequent production of highly correlated graviton squeezed states [5,6]. According to this mechanism, the energy density of gravitational origin can be estimated to be (today) of the order of 10^{-13} [7–9] (in critical units) for (present) frequencies larger than 10^{-16} Hz. The (logarithmic) energy spectrum turns out to be flat in the same interval of frequencies. The reason for this quite minute amplitude comes essentially from the measurement of the CMBR anisotropies. In fact, the tensor contribution to the CMBR anisotropies imposes a quite important bound on the maximal curvature scale at which the inflationary expansion occurred. By assuming, for example, that the inflationary phase was of de Sitter type, with typical curvature scale $H_{\rm dS}$, we have to require [10,11]

$$\frac{H_{\rm dS}}{M_P} \lesssim 10^{-6} \tag{1.1}$$

in order to be compatible with the detected level of anisotropies in the microwave sky. Constraints on the dynamical assumptions of various inflationary models can be derived on the same basis [12].

In this paper we want to explore a slightly different picture of the post-inflationary phase. Our suggestion is, in short, the following. Suppose that the inflationary phase is not immediately followed by a radiation dominated phase but by an intermediate phase whose equation of state is stiffer than radiation [i.e., $p=\gamma\rho$, with $\gamma>1$ and $\gamma=c_s^2$ (c_s is the sound velocity of the plasma)]. Then, two interesting implications can arise. On one hand the transition between the inflationary regime and the stiff regime leads to graviton spectra which slightly increase with frequency (with ''blue'' slopes) in the ultraviolet branch of the spectrum, and on the other hand, the back reaction effects associated with the GW leaving the horizon during the de Sitter phase and reentering during the stiff phase can heat up the Universe very efficiently, leading to a qualitatively new kind of reheating which one can call ''graviton reheating.''

Even though the implications of this suggestion are appealing it is certainly important to better justify some possible motivations of such a weird exercise.

First of all we can say that such a suggestion is not forbidden by any present data. Indirect evidence of the fact that the Universe might have been dominated by radiation around temperatures of the order of 0.1 MeV comes from the suc-

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cess of the simplest (homogeneous and isotropic) big-bang nucleosynthesis (BBN) scenario [13]. In the absence of any external magnetic field [14] and matter-antimatter domains [15], the light elements (^{3}He, ^{4}He, Li, D) abundances are reproduced by the BBN model provided the ratio of the baryonic charge density over the photon density is fine-tuned around 10^{-10}. However, prior to nucleosynthesis, there have been no direct tests of the thermodynamical state of the Universe and, therefore, the effective equation of state of the perfect fluid sources driving the evolution of the background geometry can be arbitrarily different than the one of a radiation dominated plasma. Moreover, there are no compelling reasons why long-range (Abelian) gauge fields should not have been present in the early Universe [15]. Owing to our ignorance of the thermodynamical state of the Universe prior to the nucleosynthesis epoch, it is possible to postulate, in the framework of a particular model, the existence of post-inflationary (decelerated) phases different than radiation.

The possibility of having post-inflationary phases whose (effective) equation of state is stiffer than radiation can also be motivated in terms of different models. It was recently argued that a stiff phase could be originated by the relaxation of the moduli towards the minimum of their nonperturbative potential [16]. A similar idea was investigated in the framework of the dilaton relaxation [17] where the resonant amplification of gauge field modes was also discussed. In fact one can argue that a regime certainly exists where the kinetic energy of the single modulus is dominant against its potential energy. Therefore, the geometry would evolve following an effective equation of state whose sound velocity is approximately equal to the speed of light. In this regime the energy density of the modulus would scale as $\rho_{\rm mod}\sim a^{-6}$. Of course intermediate situations can also be imagined so that we can say more generally that some phases with $1/3<\gamma\leqslant 1$ can occur as a result of the moduli relaxation. For example, in the context of dilaton relaxation [17], if the kinetic energy dominates we get exactly a stiff fluid model with $\gamma=1$. In fact, on theoretical grounds the dilaton field (ϕ) has the potential of going to zero (in the supersymmetric limit) as a double exponential (i.e., $V\sim\exp[-c^2\exp(-\phi)]$, with c^2 positive and with a model-dependent constant). On more physical grounds, $V(\phi)$ is believed to have one (or more) minima for some $\phi\sim\phi_{\rm min}$. When ϕ reaches curvature scale $H\sim m$ (where m is the dilaton mass) an oscillating phase begins. Depending upon the initial conditions in H_1 (the maximal curvature) the oscillating phase can be preceded by a stiff phase where the dilaton kinetic energy dominates $[\dot{\phi}^2\gg V(\phi)]$, and therefore the evolution of the dilaton in curvature will be $\phi=\phi_0+\phi_1\log[H/H_1]$ implying $\dot{\phi}^2\sim a^{-6}$.

It was recently speculated [18] (without relation to moduli relaxation) that, provided the stiff phase is long enough, interesting effects can also be expected at energy scales of the order of 100 GeV (corresponding to a curvature scale of the order $H_{ew}\sim 10^{-34}M_P$). Historically, the first one to imagine the appealing possibility of having long stiff epochs was Zeldovich [19]. At that time inflationary models had not yet been formulated and it seemed quite crucial to correctly model the quark-hadron phase transition occurring when the temperature of the Universe was of the order of the pion mass (i.e., $T_{\rm had}\sim 140$ MeV). The idea was that at this stage the equation of state of the perfect fluid sources could be stiffer than the one of radiation, namely, with $\gamma>1/3$. On physical grounds, one can easily understand that the sound velocity in a perfect fluid is likely to be smaller than the speed of light ($\gamma\leqslant 1$ in our units). In particular the stiff model of Zeldovich assumes exactly that, prior to the hadronic stage of evolution, $\gamma=1$. The fact that the speed of light equals the speed of sound also implies that the energy density of the stiff sources decreases faster than radiation.

This last feature of the stiff picture of hadronic interactions is indeed fatal for the logical consistency of the whole proposal. Actually, the existence of a hadronic phase with an effective equation of state that is stiffer than radiation also implies the production of a stochastic background of GW sharply peaked towards the Planck frequency [20]. Thus, the produced gravitational waves will back react on the geometry (effectively driven by the stiff fluid). Now, since the energy density of the high-frequency GW reentering the horizon during the stiff epoch scales similar to radiation [20,21] it was correctly concluded [20] that if a stiff phase ever existed prior to the usual radiation dominated phase, the back reaction effects associated with the production of high-frequency gravitons turned the evolution of the Universe very quickly into a radiation dominated phase. It was also shown [20,22] that there are no chances of having a stiff phase from the Planck curvature scale ($H_P\sim M_P$) down to the hadronic curvature scale ($H_{\rm had}\sim 10^{-40}M_P$).

In our context there is a crucial difference with respect to Zeldovich's suggestion and it is essentially given by Eq. (1.1). Since the curvature scale is quite minute in Planck units GW back reaction is not switched on immediately. Nonetheless, the aim of this exercise is to point out that the occurrence (and the duration) of a post-inflationary phase stiffer than radiation is not a free parameter which we can adjust in the framework of a particular model to get the desired effects. On the contrary, the duration of a stiff phase is significantly constrained by the back reaction of hard gravitons. In this sense our considerations owe very much to the pioneering works in the subject [20,22] but are applied to a different dynamical picture, where an inflationary phase is followed by a stiff phase. Our logic, in short, is the following. Let us assume, for example, that in some specific model a de Sitter (inflationary) phase is followed by a stiff phase (for example, with $p=\rho$) at some cosmic time t_1. Then the hard gravitons excited during the de Sitter phase will have typical (physical) momentum $\hbar\omega_{\rm dS}(t_1)\sim\hbar H_{\rm dS}(t_1)$. Since the Hubble distance H^{-1} deviates (during the stiff phase) from the value during the de Sitter epoch as a consequence of the change in the expansion rate the hard gravitons with $\omega\leqslant\omega_{\rm dS}$ will reenter at different times during the stiff phase. Since their effective equation of state is the one of radiation they will modify the dynamics of the stiff phase by ultimately destroying it and by turning it into a radiation dominated phase. By looking positively at this effect we could say that the graviton back reaction represents a reasonable candidate in order to implement a reasonable reheating mechanism in these classes of models. At the same time the constraints imposed by the (over)production of gravitons might be viewed as a weakness of the scenario. A related result of our analysis will be the calculation of the GW spectra produced during these types of stiff post-inflationary phases.

The plan of our paper is then the following. In Sec. II we will introduce the basic equations describing the background evolution during a stiff post-inflationary phase. Section III is devoted to the calculation of the GW spectra in these models. In Sec. IV we will discuss the back reaction effects associated with hard (nonthermal) gravitons and we will discuss the possible modification of the background evolution. Section IV contains our concluding remarks.

II. BASIC EQUATIONS

In this section we consider the simplest homogeneous and isotropic models of Friedmann-Robertson-Walker (FRW) type with line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{2.1}$$

(Greek indices run from 0 to 3, whereas Latin indices run from 1 to 3). The derivative with respect to the cosmic time t will be denoted by an overdot whereas the derivative with respect to the conformal time η will be denoted by a prime [as usual $a(\eta)d\eta = dt$]. The sign of the spatial curvature $\kappa = +1,0,-1$ corresponds to closed, flat, or open spaces. We will assume that the evolution of the geometry follows general relativity. Therefore the coupled evolution of the (perfect) fluid sources and of the geometry will be conveniently described in terms of the well known FRW equations

$$M_P^2\left[H^2 - \frac{k}{a^2}\right] = \rho - \frac{\kappa}{a^2},$$

$$M_P^2[H^2 + \dot{H}] = -\frac{1}{2}(\rho + 3p),$$

$$\dot{\rho} + 3H(\rho + p) = 0, \quad H = \frac{d\log a}{dt}, \tag{2.2}$$

where H is the Hubble parameter and we also write $M_P = (8\pi G/3)^{-1/2}$. Focusing our attention on the conformally flat case ($\kappa = 0$) we will consider models of background evolution where an inflationary phase ($\dot{a} > 0$, $\ddot{a} > 0$) is followed at some time t_1 by a decelerated phase ($\ddot{a} < 0$, $\dot{a} > 0$). In particular we will assume that the sources for $t > t_1$ will be well approximated by a barotropic equation of state

$$p = \gamma\rho, \tag{2.3}$$

with $1/3 < \gamma \leq 1$. We consider the case where $\gamma > 1$ unrealistic since this would mean that the sound velocity of the fluid is greater than the speed of light. By integrating the FRW equations (2.2) after t_1 with the closure given by Eq. (2.3) we find that

$$a(t) = a_1\left(\frac{t}{\alpha t_1}\right)^{\alpha}, \quad \rho(t) = \rho_1\left(\frac{a_1}{a}\right)^{2/\alpha} \tag{2.4}$$

with

$$\alpha = \frac{2}{3(\gamma+1)}, \quad \rho_1 = H_1^2 M_P^2, \quad a_1 = a(t_1), \tag{2.5}$$

where $H_1 = H(t_1)$ is simply the value of the Hubble parameter at the end of the inflationary phase.

We want to stress that our approach in the present section will be an effective one: since the evolution equation of the tensor modes of the geometry is essentially determined *only* by the behavior of the scalar curvature we feel free to use the simplest fluid model for the description of the background sources. This is, of course, not a limitation. The effective fluid sources can be thought of as being modeled by the energy momentum tensor of one (or more) scalar fields.

Concerning the inflationary phase we will not make any type of weird assumption about the background evolution. Indeed we will show that our considerations will only be mildly sensitive to the specific inflationary dynamics and will only (but crucially) depend upon the maximal curvature scale reached during inflation. From a purely kinematical point of view we will explore expanding inflationary epochs (i.e., $\dot{a} > 0$, $\ddot{a} > 0$) with constant or decreasing curvature (i.e., $\dot{H} \leq 0$). The de Sitter (or quasi-de Sitter) case seems to emerge quite naturally in the framework of the slow-rolling approximation [2]. It is important to point out that the de Sitter case and the power-law case will produce, respectively, either a flat or decreasing energy spectrum [12,7–9].

Having said this we will focus our attention on the case where a pure de Sitter phase is followed by a stiff phase and we will comment, where appropriate, on the other possible cases. Therefore the model we want to investigate is given (in conformal time) essentially by three phases:

$$a_i(\eta) = -\left(\frac{\eta_1}{\eta}\right), \quad \eta < -\eta_1,$$

$$a_s(\eta) = \left[\frac{(1+\beta)\eta_1 + \eta}{\beta\eta_1}\right]^{\beta}, \quad -\eta_1 < \eta < \eta_r,$$

$$a_r(\eta) = \frac{\beta\eta + (\beta+1)\eta_1 - (\beta-1)\eta_r}{[\beta\eta_1]^{\beta}[\eta_r + (\beta+1)\eta_1]^{1-\beta}}, \quad \eta > \eta_r. \tag{2.6}$$

The subscripts i, s, and r simply stand for inflationary, stiff, and radiation dominated phases. An important feature of Eq. (2.6) is the continuity of the scale factors (and of their first conformal time derivatives) in the matching points η_1 and η_r. Notice also that the generic exponent β specifying the dynamics during the stiff phase is trivially related to the previously introduced α and γ parameters, namely,

$$\beta = \frac{\alpha}{1-\alpha} = \frac{2}{3\gamma+1}. \tag{2.7}$$

Of course the three phases specified by Eq. (2.6) are usually complemented by the transition to matter domination occurring at $\eta_{\rm dec}$. For $\eta > \eta_{\rm dec}$ the scale factor evolves parabolically [i.e., $a_m(\eta) \sim \eta^2$] in the matter epoch. Our main goal in this paper will be to study the gravitational wave spectra in these models and to study which kind of constraints arise by changing H_1 and γ. Notice that the time of the inflation-stiff phase transition is $\eta_1 = [a_i(t_1) H_i(t_1)]^{-1}$.

III. GRAVITATIONAL WAVE SPECTRA FROM STIFF PHASES

The evolution of the scalar, vector, and tensor fluctuations of a given background geometry can be directly discussed by perturbing (to second order in the amplitude of the fluctuations) the Einstein-Hilbert action. An important property of the metric perturbations in FRW backgrounds of the type defined in Eq. (2.1) is that scalar, vector, and tensor modes are decoupled [23] (this feature does not hold in the case of anisotropic background geometries [24]). This fact means that by defining the fluctuations of the background metric $\overline{g}_{\mu\nu}$ as

$$g_{\mu\nu}(\vec{x},\eta) = \overline{g}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\vec{x},\eta), \tag{3.1}$$

the metric fluctuation can be formally written as

$$\delta g_{\mu\nu}(\vec{x},\eta) = \delta g^{(S)}_{\mu\nu}(\vec{x},\eta) + \delta g^{(V)}_{\mu\nu}(\vec{x},\eta) + \delta g^{(T)}_{\mu\nu}(\vec{x},\eta), \tag{3.2}$$

where S, V, T stand respectively for, scalar, vector, and tensor modes. This classification refers to the way in which the fields from which $\delta g_{\mu\nu}(\vec{x},\eta)$ are constructed, change under three-dimensional (spatial) coordinate transformations on the constant (conformal)-time hypersurface. Since $\delta g_{\mu\nu}(\vec{x},\eta)$ (being a symmetric four-dimensional tensor of rank 2) has ten independent components the scalar, vector, and tensor modes will be parametrized by ten independent space-time functions. More specifically, scalar perturbations will be parametrized by four independent scalar functions and vector perturbations by two divergenceless three-dimensional vectors (equivalent to four independent functions). Pure tensor modes of the metric (corresponding to physical gravitational waves propagating in a homogeneous and isotropic background) can be constructed using a symmetric three tensor h_{ij} satisfying the constraints

$$h_{0\mu} = 0, \quad h_i^i = 0, \quad \nabla^j h_{ij} = 0 \tag{3.3}$$

(∇_i denotes the covariant derivative with respect to the three-dimensional metric). Notice that with our conventions $\delta g^{(T)}_{\mu\nu} = h_{\mu\nu}$ and $\delta g^{\mu\nu}_{(T)} = -h^{\mu\nu}$. The line element perturbed by the tensor modes can then be written as

$$ds^2 = a^2(\eta)[d\eta^2 - (\gamma_{ij} + h_{ij}) dx^i dx^j] \tag{3.4}$$

(where γ_{ij} is the spatial background metric).

Summarizing, we have four functions for scalars, four functions for vectors, and two functions (the two polarizations of h_{ij}) for the tensors. In total there are ten independent degrees of freedom describing the fluctuations as required by the tensor properties of the original (unperturbed) metric. Moreover the conditions expressed in Eq. (3.3) imply that the two physical (independent) polarizations of h_{ij} do not contain any pieces which transform as scalars or vectors under three-dimensional rotations. As a consequence of its definition, h_{ij} is directly invariant under infinitesimal coordinate transformations preserving the tensorial character of the fluctuations [4,23].

In order to obtain the evolution equation of the metric fluctuations there are at least two different procedures. First of all one could think to perturb (to first order in the metric fluctuations) the Einstein equations. On the other hand one could also perturb the Einstein-Hilbert action

$$S = -\frac{1}{6 l_P^2} \int d^4x \sqrt{-g} R, \quad R = g^{\alpha\beta} R_{\alpha\beta},$$
$$g = \det[g_{\mu\nu}], \quad l_P = M_P^{-1} \tag{3.5}$$

to second order in the amplitude of the metric fluctuations [25].

It is convenient to notice that the two procedures are certainly equivalent but the perturbation of the action provides more information since it allows us to isolate (up to total derivative terms) the normal modes of oscillation of the system which one might want to normalize to the value of the quantum-mechanical fluctuations. By perturbing the action given in Eq. (3.5) to second order in the amplitude of the tensor fluctuations we obtain that, up to total derivatives [24],

$$\delta^{(2)} S^{(T)} = \frac{1}{24 l_P^2} \int d^4x \sqrt{-\overline{g}} [\partial_\alpha h_{ij} \partial_\beta h^{ij} \overline{g}^{\alpha\beta}] \tag{3.6}$$

[we remind the reader that in this and in the following formulas the shift from upper to lower spatial indices (and vice versa) is done by using the spatial background metric γ_{ij} and its inverse].

For a wave moving in the $x^3 = z$ direction in our coordinates one has $h_\oplus = h_1^1 = -h_2^2$ and $h_\otimes = h_1^2 = h_2^1$ and the perturbed action becomes

$$\delta^{(2)} S^{(T)} = \frac{1}{12 l_P^2} \int d^4x \sqrt{-\overline{g}} [\partial_\alpha h_\oplus \partial_\beta h_\oplus \overline{g}^{\alpha\beta} + \partial_\alpha h_\otimes \partial_\beta h_\otimes \overline{g}^{\alpha\beta}]. \tag{3.7}$$

By now varying the perturbed action we get the evolution equation for each polarization:

$$\ddot{h}_\oplus + 3H\dot{h}_\oplus - \nabla^2 h_\oplus = 0, \quad \ddot{h}_\otimes + 3H\dot{h}_\otimes - \nabla^2 h_\otimes = 0. \tag{3.8}$$

From Eq. (3.7) it is also possible to deduce the form of the canonical normal modes, namely, those modes whose Lagrangian reduces to the Lagrangian of two minimally coupled scalar fields (in flat space) with time-dependent mass terms. Defining

$$\mu_{\oplus}=\frac{ah_{\oplus}}{\sqrt{6}l_P},\quad \mu_{\otimes}=\frac{ah_{\otimes}}{\sqrt{6}l_P},\tag{3.9}$$

we can write, from Eq. (3.7), the action for the canonical normal modes

$$\delta^{(2)}S^{(T)}=\int d^3x d\eta \;\; L,\tag{3.10}$$

where (always up to total derivatives)

$$L=\frac{1}{2}[\eta^{\alpha\beta}\partial_\alpha\mu_\oplus\partial_\beta\mu_\oplus+\eta^{\alpha\beta}\partial_\alpha\mu_\otimes\partial_\beta\mu_\otimes + (\mathcal{H}^2+\mathcal{H}')(\mu_\oplus{}^2+\mu_\otimes{}^2)]\tag{3.11}$$

($\eta_{\alpha\beta}=\mathrm{diag}(1,-1,-1,-1)$ is the flat space-time metric and $\mathcal{H}=[\log a]'$). From Eq. (3.11) we can derive the evolution equations of the canonical normal modes by taking the functional variation with respect to $\mu_\otimes$ and $\mu_\oplus$ with the result that

$$\mu_\otimes''-\nabla^2\mu_\otimes-\frac{a''}{a}\mu_\otimes=0,\quad \mu_\oplus''-\nabla^2\mu_\oplus-\frac{a''}{a}\mu_\oplus=0.\tag{3.12}$$

The normal modes obtained in Eq. (3.19) can be now canonically quantized. First of all we define the canonical momenta

$$\pi_\oplus=\frac{\partial L}{\partial\mu_\oplus'},\quad \pi_\otimes=\frac{\partial L}{\partial\mu_\otimes'},\tag{3.13}$$

leading to the Hamiltonian

$$Q=\int d^3x[\pi_\oplus\mu_\oplus'+\pi_\otimes\mu_\otimes'-L].\tag{3.14}$$

We then impose the (equal time) commutation relations between the corresponding field operators:

$$\begin{aligned}&[\hat{\mu}_\oplus(\vec{x},\eta),\hat{\pi}_\oplus(\vec{y},\eta)]=i\delta^3(\vec{x}-\vec{y}),\\ &[\hat{\mu}_\otimes(\vec{x},\eta),\hat{\pi}_\otimes(\vec{y},\eta)]=i\delta^3(\vec{x}-\vec{y}).\end{aligned}\tag{3.15}$$

We want to remind the reader that the way we obtained the Hamiltonian is quite naive. In fact there is a formally more correct way of getting the Hamiltonian for the perturbations [26]. One should, in fact, start with the Hamiltonian in superspace, fix the background, and then obtain the Hamiltonian for the perturbations together with four (independent) constraints (two for scalar perturbations and two for vector perturbations). Therefore, in a fully consistent Hamiltonian approach to perturbations there are no constraints arising from the tensor modes. This is the reason why our naive approach leads to the same result of the Hamiltonian formalism with respect to the tensor modes. In the case of scalar fluctuations, however, our approach would be less correct since, in the scalar case, the off-diagonal components of the Einstein equations receive a contribution leading to constraints. In order to get the Hamiltonian for the scalar modes (and the related algebra of the constraints) the (Hamiltonian) superspace approach is more compelling. We point out that, in any case, in our present approach it is also possible to get the correct result for the Hamiltonian of the scalar modes by inserting, in the perturbed action of the scalar fluctuations, the constraint equation arising from the $(0i)$ components of the Einstein equations [27].

Promoting the classical Hamiltonian quantum-mechanical operator the evolution equations for the field operators become, in the Heisenberg representation,

$$\begin{aligned}&i\hat{\mu}_\oplus'=[\hat{\mu}_\oplus,\hat{Q}],\quad i\hat{\pi}_\oplus'=[\hat{\pi}_\oplus,\hat{Q}],\\ &i\hat{\mu}_\otimes'=[\hat{\mu}_\otimes,\hat{Q}],\quad i\hat{\pi}_\otimes'=[\hat{\pi}_\otimes,\hat{Q}].\end{aligned}\tag{3.16}$$

Notice that (as it has to be) the evolution equations in the Heisenberg representation are exactly identical (for the field operators) to the ones derived for the classical fields in Eq. (3.12) once the explicit expressions of the conjugated momenta (i.e., $\hat{\pi}_\oplus=\hat{\mu}_\oplus'$ and $\hat{\pi}_\otimes=\hat{\mu}_\otimes'$) are inserted back into Eq. (3.16). We also point out that the Hermitian Hamiltonian of Eq. (3.14) is quadratic in the field operators and belongs to a general class of time-dependent Hamiltonians widely used in quantum optics [5,28] in the context of the parametric amplification of the vacuum fluctuations of the electromagnetic field through laser beams. The same discussion of the Heisenberg picture can be easily translated to the Schrödinger picture where the final state of evolution is a many particle state unitarily connected to the initial vacuum (a so-called squeezed state [5,28]). Having fixed this standard notation we can expand the operators in Fourier integrals

$$\begin{aligned}\hat{\mu}_\oplus&=\frac{1}{(2\pi)^{3/2}}\int d^3k[\mu_\oplus(k,\eta)\hat{a}_\oplus(\vec{k})e^{i\vec{k}\cdot\vec{x}}+\mu_\oplus^*(k,\eta)\hat{a}_\oplus^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}}],\\ \hat{\mu}_\otimes&=\frac{1}{(2\pi)^{3/2}}\int d^3k[\mu_\otimes(k,\eta)\hat{a}_\otimes(\vec{k})e^{i\vec{k}\cdot\vec{x}}+\mu_\otimes^*(k,\eta)\hat{a}_\otimes^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}}]\end{aligned}\tag{3.17}$$

(recall that, in our notation, $\hat{h}_{\oplus,\otimes}(\vec{x},\eta)=\sqrt{6}[l_P/a]\hat{\mu}_{\oplus,\otimes}(\vec{x},\eta)$). From Eq. (3.15) $\hat{a}_\oplus$ and $\hat{a}_\otimes$ obey the following commutation relations:

$$\begin{aligned}&[\hat{a}_\oplus(\vec{k}),\hat{a}_\oplus^\dagger(\vec{k}')]=\delta^{(3)}(\vec{k}-\vec{k}'),\\ &[\hat{a}_\otimes(\vec{k}),\hat{a}_\otimes^\dagger(\vec{k}')]=\delta^{(3)}(\vec{k}-\vec{k}'),\\ &[\hat{a}_\oplus(\vec{k}),\hat{a}_\otimes(\vec{k}')]=0.\end{aligned}\tag{3.18}$$

Therefore the evolution equations for the Fourier amplitudes $\mu_{\oplus}(k,\eta)$ and $\mu_{\otimes}(k,\eta)$ become

$$\mu''_{\oplus,\otimes}+[k^2-f(\eta)]\mu_{\oplus,\otimes}=0, \quad f(\eta)=\frac{a''}{a}\equiv a^2(\dot{H}+2H^2) \tag{3.19}$$

[where, for the Fourier amplitudes, we define $\mu(k,\eta) = ah(k,\eta)$].

By taking the functional derivative of the action reported in Eq. (3.6) with respect to the background metric we indeed find an effective energy-momentum tensor of the fluctuations which reads

$$\tau_{\mu\nu}=\frac{1}{6\ \ l_P^2}\bigg[\partial_\mu h_{\oplus}\partial_\nu h_{\oplus}+\partial_\mu h_{\otimes}\partial_\nu h_{\otimes} -\frac{1}{2}\bar{g}_{\mu\nu}(\bar{g}^{\alpha\beta}\partial_\alpha h_{\oplus}\partial_\beta h_{\oplus}+\bar{g}^{\alpha\beta}\partial_\alpha h_{\otimes}\partial_\beta h_{\otimes})\bigg]. \tag{3.20}$$

If we define the vacuum state vector $|0_{\oplus}0_{\otimes}\rangle$ which is annihilated by $\hat{a}_{\oplus}$ and $\hat{a}_{\otimes}$ we obtain that the energy density of the produced gravitons will be given by

$$\rho_{\rm GW}(\eta)=\langle 0_{\otimes}0_{\oplus}|\tau_{oo}|0_{\oplus}0_{\otimes}\rangle =\frac{1}{16\pi^3\ \ a^2}\int d^3k\{|h'_{\oplus}(k,\eta)|^2+|h'_{\otimes}(k,\eta)|^2 +k^2[|h_{\oplus}(k,\eta)|^2+|h_{\otimes}(k,\eta)|^2]\}. \tag{3.21}$$

In order to compute the GW spectra produced by the transition of the background from an inflationary phase to a decelerated stiff phase we have to solve the evolution equation (3.19). From Eq. (3.19) we clearly see that the evolution equation of the (tensor) normal modes of the geometry is determined not only by H^2 but also by $\dot{H}$. Therefore, depending upon the sign of $\dot{H}$ different inflationary models will give different (large scale) spectral distributions of the amplified gravitational waves.

In the three phases defined in Eq. (2.6) the evolution equation (3.19) reads

$$\mu''+\left[k^2-\frac{2}{\eta^2}\right]\mu=0, \quad \eta<-\eta_1,$$

$$\mu''+\left[k^2-\frac{\beta(\beta-1)}{[\eta+(\beta+1)\eta_1]^2}\right]\mu=0, \quad -\eta_1<\eta<\eta_r,$$

$$\mu''+k^2\mu=0, \quad \eta_{\rm dec}<\eta<\eta_r \tag{3.22}$$

(notice that we dropped the subscript referring to each polarization). The solution of the evolution equations in the three regions can be written in terms of Hankel functions [29,30]. Since we are dealing with Fourier amplitudes of the normal modes of oscillation we normalize them directly to the quantum mechanical noise level ($\sim 1/\sqrt{k}$) for $\eta<-\eta_1$:

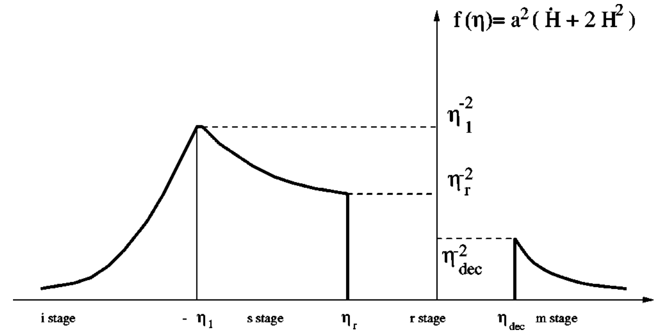

FIG. 1. The "effective potential" a''/a appearing in the Schrödinger-like equation (3.19) is reported as a function of the conformal time coordinate. In the standard scenario, the intermediate (stiff) phase is absent and, therefore, all the modes k will reenter during the radiation epoch leading (at high frequencies) to the usual Harrison-Zeldovich (flat) energy spectrum. In our case the modes reentering during the radiation dominated era ($k<\eta_r^{-1}$) will always have a flat energy spectrum. At the same time, some modes going under the potential barrier during the de Sitter phase (i stage) and reentering the horizon during the stiff phase (s stage) will lead to a modification of the high-frequency branch of the spectrum which will become mildly increasing [see Eqs. (3.30),(3.31), and Eq. (3.32)]. We stress that at large (10^{-18} Hz$<\omega<10^{-16}$ Hz) and intermediate (10^{-16} Hz$<\omega<\omega_r$) (present) physical frequencies, the spectrum does not change with respect to the usual results [12,7–9] since the corresponding modes will not "feel" the presence of the stiff phase.

$$\mu_I(u)=\frac{1}{\sqrt{2k}}\sqrt{u}H^{(2)}_{3/2}(u), \quad \eta<-\eta_1,$$

$$\mu_{\rm II}(v)=\frac{1}{\sqrt{2k}}\sqrt{v}[b_+H^{(2)}_\nu(v)+b_-H^{(1)}_\nu(v)], \quad -\eta_1<\eta<\eta_r,$$

$$\mu_{\rm III}(u)=\frac{1}{\sqrt{2k}}[c_+e^{-iu}+c_-e^{iu}], \quad \eta_{\rm dec}<\eta<\eta_r, \tag{3.23}$$

where

$$2\nu=|2\beta-1|, \quad u=k\eta, \quad v=k[\eta+(\beta+1)\eta_1]. \tag{3.24}$$

Notice that the Wronskian normalization of the solutions imposes that $|b_+|^2-|b_-|^2=1$ and $|c_+|^2-|c_-|^2=1$. See also Fig. 1 for a pictorial description of the amplification process. By computing the coefficients b_- and c_- we can have an estimate of the energy spectrum of the produced gravitons reentering the horizon, respectively, after η_1 (i.e., right after the completion of the inflationary phase) and after η_r (i.e., immediately after the beginning of the radiation dominated epoch). Notice that η_r appearing in Eq. (3.23) is not a free parameter since the duration of the stiff phase ending in η_r is determined essentially by the back reaction of the modes which reentered right after η_1. In order to further clarify this point we have to compute the GW energy spectra produced right after η_1. The energy density of the produced gravitons per logarithmic interval of longitudinal momentum can be obtained from Eq. (3.21):

$$\rho(\omega,t)=\frac{d\rho_{GW}(\omega,t)}{d\log\omega}, \tag{3.25}$$

where $\omega=k/a$. Sometimes $\rho(\omega,t)$ is called logarithmic energy spectrum. Notice that one can define the energy density of the gravitational wave background (in a semiclassical description) through the energy-momentum pseudotensor of the amplified tensor modes. In the approach of the energy-momentum pseudotensor the quantum-mechanical expectation values are replaced by ensemble averages over a distribution of stochastic variables whose two-point function has to be specified separately by requiring that each Fourier amplitude of the tensor modes is independent of the others in momentum space. This ''stochastic'' condition on the Fourier amplitudes is the result, in a fully quantum-mechanical approach, of the unitarity of the graviton production process driven by the Hermitian Hamiltonian (3.14). We stress that if our initial state is not the vacuum (but, for example, a non-pure state characterized by a specific density matrix [28]), the expression given in Eq. (3.21) for the averaged energy density will also be different.

From Eq. (3.21) and using the definition (3.25) we have that in each phase of the model the energy spectra can be written as

$$\rho(\omega,t)\simeq\frac{1}{2\pi^2}H_1^4\left(\frac{\omega}{\omega_1}\right)^4|b_-|^2\left[\frac{a_1}{a}\right]^4, \quad -\eta_1<\eta<\eta_r,$$

$$\rho(\omega,t)\simeq\frac{1}{2\pi^2}H_1^4\left(\frac{\omega}{\omega_1}\right)^4|c_-|^2\left[\frac{a_1}{a}\right]^4, \quad \eta>\eta_r. \tag{3.26}$$

In order to compute the coefficients $|b_-|$ and $|c_-|$ we use the sudden approximation, namely, we will mainly consider the amplification of the modes $k\eta_1\lesssim 1$ [31]. A possible technique is to match, in η_1 and η_r, the exact solutions given by Eq. (3.23) and their first derivatives. Therefore, imposing the conditions

$$\mu_{I}(u_1)=\mu_{II}(v_1), \quad \mu_{I}'(u_1)=\mu_{II}'(v_1),$$

$$\mu_{II}(v_r)=\mu_{III}(u_r), \quad \mu_{II}'(v_r)=\mu_{III}'(u_r), \tag{3.27}$$

we obtain an expression for the amplification coefficients which we evaluate in the sudden approximation

$$|b_-|\simeq\left(\frac{\omega}{\omega_1}\right)^{-3/2-\nu}, \quad \omega_r<\omega<\omega_1,$$

$$|c_-|\simeq\left(\frac{\omega}{\omega_r}\right)^{-1/2}\left(\frac{\omega}{\omega_1}\right)^{-3/2}\left(\frac{\omega_1}{\omega_r}\right)^{\nu}, \quad \omega_{dec}<\omega<\omega_r \tag{3.28}$$

(notice that $u_{1,r}=k\eta_{1,r}$ and $v_{1,r}=k[\eta_{1,r}+(\beta+1)\eta_1]$). These expressions are correct as long as the effective equation of state parametrizing the evolution of the sources in the intermediate phase has $\gamma<1$. For $\gamma=1$ (corresponding to $\beta=\frac{1}{2}$, $\alpha=\frac{1}{3}$ and $\nu=0$), the solution of Eq. (3.22) in the intermediate phase [i.e., what we call $\mu_{II}(v)$ in Eq. (3.23)] is given in terms of $H_0^{(1,2)}(v)$ and, therefore, the amplification coefficients get logarithmically corrected (in the sudden approximation), and the final result is

$$|b_-|\simeq\left(\frac{\omega}{\omega_1}\right)^{-3/2}\log\left(\frac{\omega_1}{\omega_r}\right), \quad \omega_r<\omega<\omega_1,$$

$$|c_-|\simeq\left(\frac{\omega}{\omega_r}\right)^{-1/2}\left(\frac{\omega}{\omega_1}\right)^{-3/2}\log\left(\frac{\omega_1}{\omega_r}\right), \quad \omega_{dec}<\omega<\omega_r. \tag{3.29}$$

Notice that $\omega_{dec}\simeq 10^{-16}$ Hz is the frequency corresponding to η_{dec}, namely, to the moment of the transition to the matter-dominated epoch. Of course the gravitational wave spectra will have a further (fourth) branch corresponding to the modes reentering the horizon during the radiation dominated epoch [32]. This infrared branch of the spectrum concerns frequencies $\omega_0<\omega<\omega_{dec}$, where $\omega_0\simeq 10^{-18}$ Hz is the frequency corresponding to the present horizon. It is well known that in the infrared branch the spectrum turns out to be steeply decreasing as ω^{-2} [9,32]. Since the purpose of our paper is the investigation of the effects of the produced gravitons during the stiff phase we will neglect, in the following considerations, the infrared branch.

By inserting the obtained expressions of the amplification coefficients in the expression for the energy spectrum of Eq. (3.26) we get the main result of this section

$$\rho(\omega,t)\sim\frac{1}{2\pi^2}H_1^4\left(\frac{\omega}{\omega_1}\right)^{1-2\nu}\left[\frac{a_1}{a}\right]^4, \quad \omega_r<\omega<\omega_1,$$

$$\rho(\omega,t)\sim\frac{1}{2\pi^2}H_1^4\left(\frac{\omega_r}{\omega_1}\right)^{1-2\nu}\left[\frac{a_1}{a}\right]^4, \quad \omega_{dec}<\omega<\omega_r. \tag{3.30}$$

Again, this expression is logarithmically corrected in the $\gamma\to 1$ limit:

$$\rho(\omega,t)\sim\frac{1}{2\pi^2}H_1^4\left(\frac{\omega}{\omega_1}\right)\log^2\left(\frac{\omega}{\omega_1}\right)\left[\frac{a_1}{a}\right]^4, \quad \omega_r<\omega<\omega_1,$$

$$\rho(\omega,t)\sim\frac{1}{2\pi^2}H_1^4\left(\frac{\omega_r}{\omega_1}\right)\log^2\left(\frac{\omega_r}{\omega_1}\right)\left[\frac{a_1}{a}\right]^4, \quad \omega_{dec}<\omega<\omega_r. \tag{3.31}$$

A few comments are in order concerning Eqs. (3.30) and (3.31). First of all we can notice that the energy spectra are scale invariant for $\omega<\omega_r$. This branch of the spectrum corresponds to modes which went out of the horizon during the initial de Sitter phase and reentered during the radiation dominated phase. There is, therefore, no surprise for this behavior which corresponds to the usual Harrison-Zeldovich spectrum of the stochastic GW backgrounds. Moreover, we point out that for $\omega_r \to \omega_1$ (i.e., pointlike stiff phase) we reproduce the usual results well known in the context of the theory of graviton production when a de Sitter phase is suddenly followed by a radiation dominated phase [12,9,32]. For modes leaving the horizon during the de Sitter phase and reentering during the stiff phase ($\omega_r < \omega < \omega_1$) we get the curious result that the energy spectrum increases as a function of the frequency. This feature of our result can be easily understood by bearing in mind that the ultraviolet branch of the spectrum goes as

$$\left(\frac{\omega}{\omega_1}\right)^{1-2\nu}, \quad \nu = \frac{3(1-\gamma)}{2(3\gamma+1)}. \tag{3.32}$$

If $\frac{1}{3} < \gamma \leq 1$ we have that $2\nu < 1$ and the spectrum always mildly increases with maximal slope (ω/ω_1) (for $\gamma = 1$). The minimal slope corresponds to the case $\gamma = 1/3$ (flat case) where the flat spectrum is recovered for the entire (present) frequency range. This peculiar behavior is really not so strange. This peculiar feature of stiff models was indeed noticed long ago [20]. In our case the only difference is that prior to the stiff phase there is a de Sitter phase and, therefore, the calculation of the amplification coefficients involves a further transition. This further transition modifies the high-energy behavior of the spectrum which still increases but more mildly if compared to the case where the de Sitter phase was absent [20]. If the inflationary phase is not de Sitter but power law (i.e., $\dot{H} < 0$) or superinflationary [6] ($\dot{H} > 0$) the spectra of the modes reentering in the radiation epoch will be crucially modified and their amplitudes will always be subjected to the large scale constraint ($H_1 \lesssim 10^{-6} M_P$) provided that the inflationary phase is either de Sitter–like or power law. In the case of superinflation (leading to increasing energy spectra), the nucleosynthesis constraint will always be the most stringent one [32,33] since it involves the integrated energy density and is applied at all frequencies.

The background energy decreases during the stiff phase as $H_1^2 M_P^2 (a_1/a)^{3(\gamma+1)}$, whereas the radiation stored in GW decreases as $(a_1/a)^4$ and therefore, at some stage, the graviton radiation will become dominant. To precisely compute this moment we have to integrate the graviton spectrum over all the modes, insert it back into the Einstein equations and, finally, solve the modified Einstein equations. By defining $\epsilon = H_1/M_P$, a simple argument based on the calculations reported in the previous section shows that, since the energy spectra are increasing, the most significant contribution of the hard gravitons to the energy density (integrated over the whole spectrum) will occur for $\omega \sim \omega_1$. This energy density will turn the stiff background into radiation at a critical value of the scale factor $a_r \sim \epsilon^{2/(1-3\gamma)} a_1$ and the stiff fluid will correspondingly turn into radiation (i.e., $\gamma \to 1/3$). Taking into account now that during the stiff phase $a(t) \sim t^{2/[3(\gamma+1)]}$ we have that, according to our estimate, the back reaction effects will become significant at a curvature scale $H_r \sim \epsilon^{(6\gamma+2)/(3\gamma-1)} M_P$. For example, if we take $\gamma = 1$, the curvature scale at which the transition to radiation takes place is $H_r \sim \epsilon^4 M_P$. In frequency, the length of the stiff phase (with $\gamma = 1$) will then be $\omega_r/\omega_1 \sim \eta_1 a_1/\eta_r a_r \sim \epsilon^3$. Thus, if one wants to reach (within a stiff phase with $\gamma = 1$) the hadronic curvature scale $H_{\rm had} \sim 10^{-40} M_P$ we should fine-tune $\epsilon \sim 10^{-10}$.

The maximal scale H_1 is certainly constrained. If a de Sitter phase is immediately followed by a radiation dominated phase, the GW energy spectrum is flat for $\omega_{\rm dec} < \omega < \omega_1$ (notice that, today, $\omega_{\rm dec} \sim 10^{-16}$ Hz and $\omega_1 \sim 10^{11}\sqrt{\epsilon}$ Hz). For $\omega_0 < \omega < \omega_{\rm dec}$ the spectrum decreases and, therefore, $\epsilon \lesssim 10^{-6}$ which is exactly the bound reported in Eq. (1.1) coming from the tensor contribution to the CMBR anisotropy.

If the de Sitter phase is followed by a stiff phase the spectra grow for $\omega_r < \omega < \omega_1$. Take, for instance, the case $\gamma = 1$ where $\rho(\omega,t) \sim H_1^4(\omega/\omega_1)$ up to logarithmic corrections. In this branch, since the spectrum increases, the most significant constraint comes from the bound energy density in relativistic degrees of freedom at the nucleosynthesis epoch [32,33]. Thus the bounds on ϵ might be a bit different (even if not by much due to the very mild increase of the spectral energy density). Today, in the range $\omega_{\rm dec} < \omega < \omega_r$ the spectral energy density (in critical units) would be $\Omega_{\rm GW}(\omega,t) \sim 10^{-4}\epsilon^2(\omega_r/\omega_1)$. Now the large scale bound imposes that in this phase $\Omega_{\rm GW} \lesssim 10^{-13}$. Taking into account now that $\omega_r/\omega_1 \sim \epsilon^3$ (for the case $\gamma = 1$) we have that $\Omega_{\rm GW}(\omega,t) \sim \epsilon^5$. Imposing $\epsilon \lesssim 10^{-3}$ now we get that, for $\omega_{\rm dec} < \omega < \omega_r$, $\Omega_{\rm GW}(\omega,t) < 10^{-14}$. The argument can be easily extended to the entire class of stiff models (i.e., $1/3 < \gamma < 1$).

In conclusion, the possibility of having a stiff post-inflationary phase necessarily implies a reheating driven by gravitational waves reentering the horizon during the stiff phase. In order to have a significantly long stiff phase, however, fine-tuning is strictly required making these models perhaps less attractive. We will elaborate on this point in the next section.

IV. BACK REACTION EFFECTS

In order to estimate the length of the transition between the stiff phase and the radiation dominated phase induced by the hard gravitons, we rewrite Eq. (3.19) in a slightly different form, namely [22],

$$\left[\frac{d^2}{d\tau^2} + \Omega_k^2(\tau)\right] h_{\oplus,\otimes} = 0, \quad d\eta = \frac{dt}{a} = a^2 d\tau, \tag{4.1}$$

where $\Omega_k(\tau) = \sqrt{-g}\,\omega = ka^2$ and $h = a^{-1}\mu$. A formal solution to this equation can be written as

$$h(k,\tau) = \frac{1}{\sqrt{2\Omega_k}}[C_+ h_-(k,\tau) + C_- h_+(k,\tau)], \tag{4.2}$$

with

$$h_{\pm}(k,\tau)=\exp\left[\pm i\int \Omega_k d\tau\right] \tag{4.3}$$

(notice that C_+ and C_- are complex functions of τ; thanks to the Wronskian normalization condition we also have that $|C_+|^2-|C_-|^2=1$). By inserting the solution given in Eq. (4.2) back into Eq. (4.1) we can obtain an evolution equation for the two, time-dependent coefficients $C_\pm$ [34]

$$\frac{dC_+}{d\tau}=\frac{1}{2}\frac{d\log\Omega_k}{d\tau}h_+^2C_-, \quad \frac{dC_-}{d\tau}=\frac{1}{2}\frac{d\log\Omega_k}{d\tau}h_-^2C_+ \tag{4.4}$$

(from now on we will drop the subscript referring to the two polarizations). We are now going to solve these equations for the modes $k\eta_1 \lesssim 1$. It is easy to show that this sudden approximation used in the previous section corresponds, in the language of Eq. (4.1), to the small $\int^{\tau_1}d\tau\Omega_k(\tau)$ limit. In fact

$$\int^{\tau_1}\Omega_k(\tau)d\tau=\int^{\tau_1}ka^2d\tau\equiv\int^{\eta_1}kd\eta\sim k\eta_1 \tag{4.5}$$

(the last two equalities follow from the definition of $d\tau$ in terms of the conformal time coordinate $d\tau=d\eta/a^2$).

In this approximation we can expand the $h_\pm$ appearing in Eq. (4.4) and we find, to first order in $\int d\tau\Omega_k$,

$$\begin{aligned}\frac{dC_+}{d\tau}&=\frac{1}{2}\frac{d\log\Omega_k}{d\tau}\left[1+2i\int \Omega_k d\tau\right]C_-,\\ \frac{dC_-}{d\tau}&=\frac{1}{2}\frac{d\log\Omega_k}{d\tau}\left[1-2i\int \Omega_k d\tau\right]C_+.\end{aligned} \tag{4.6}$$

By linearly combining the two previous equations we find

$$\begin{aligned}\frac{d}{d\tau}[C_++C_-]&=\frac{1}{2}\frac{d\log\Omega_k}{d\tau}[C_++C_-]+\cdots,\\ \frac{d}{d\tau}[C_+-C_-]&=-\frac{1}{2}\frac{d\log\Omega_k}{d\tau}[C_+-C_-]+\cdots .\end{aligned} \tag{4.7}$$

In Eq. (4.7) the ellipses stand for other terms which are of higher order in $\int^{\tau_1}\Omega_k(\tau)d\tau\sim k\eta_1$ and which are negligible in the sudden approximation. The solution to Eq. (4.7) can easily be found in terms of two (arbitrary) complex coefficients

$$C_++C_-=2Q_1\sqrt{\Omega_k}, \quad C_+-C_-=2Q_2\frac{1}{\sqrt{\Omega_k}}. \tag{4.8}$$

As we mentioned previously, the Wronskian condition imposes that $|C_+|^2-|C_-|^2=1$. This last condition has to hold order by order in $k\eta_1$ and, therefore, inserting the $C_\pm$ of Eq. (4.8) into $|C_+|^2-|C_-|^2=1$ we obtain a condition on Q_1 and Q_2 valid to first order in $k\eta_1$:

$$C_+=\left[Q_1\sqrt{\Omega_k}+\frac{Q_2}{\sqrt{\Omega_k}}\right], \quad C_-=\left[Q_1\sqrt{\Omega_k}-\frac{Q_2}{\sqrt{\Omega_k}}\right],$$
$$Q_1^*Q_2+Q_1Q_2^*=\frac{1}{2}. \tag{4.9}$$

Since the amount of the amplification of the GW has essentially been determined in this approach, by $|C_-|^2$ we fix Q_1 and Q_2 by requiring that at the time η_1 (or t_1) $|C_-(k,t_1)|=0$. Thus, from Eq. (4.9) we get that

$$|Q_1|=\frac{1}{2\sqrt{\Omega_k(t_1)}}, \quad |Q_2|=\frac{\sqrt{\Omega_k(t_1)}}{2}, \quad \Omega_k(t_1)=a_1^2k. \tag{4.10}$$

Using Eq. (4.2) and summing the polarizations we get that

$$\begin{aligned}\rho_{\rm GW}(t)&=\frac{1}{8\pi^3}\int d^3\omega\,\omega[|C_+(\omega,t)|^2+|C_-(\omega,t)|^2]\\ &=\frac{1}{8\pi^3}\int \omega d^3\omega[2|C_-(\omega,t)|^2+1]\end{aligned} \tag{4.11}$$

(where in the last equality we used the Wronskian normalization condition). To the lowest order in $k\eta_1$, using Eqs. (4.9) and (4.8) and expressing all the quantities in terms of the corresponding physical momenta ω, we obtain

$$\rho_{\rm GW}(t)=\frac{1}{16\pi^3}\int \omega d^3\omega\left[\frac{\Omega_\omega(t)}{\Omega_\omega(t_1)}+\frac{\Omega_\omega(t_1)}{\Omega_\omega(t)}\right]. \tag{4.12}$$

Now we want to compute how much energy is present at a generic time t inside the horizon during the stiff phase. First of all we will outline the formal solution of the problem and secondly we will do an explicit calculation.

The total energy density in GW at a generic time t will be given by the explicit momentum integral indicated in Eq. (4.12). This is however not the end of the story. At any given time different GWs will reenter during the stiff phase and therefore the total energy stored in GWs which should appear at the right hand side of the Einstein equations is the sum over all waves reentering at different times after t_1. This means that the quantity we should compute is [22] the total energy density of the gravitational waves reentering at some generic time t during the stiff phase. We then have that the total energy density of the graviton background is given by the modes reentering at t summed to the (redshifted) energy density of those modes which reentered during the period $t_1<t'<t$:

$$\rho_{\rm tot}(t)=-\int_{t_1}^{t}\left[\frac{a(t')}{a(t)}\right]^4\left[\frac{\partial\rho_{\rm GW}(\omega_m(t'),t')}{\partial\omega_m}\right]_a\frac{\partial\omega_m}{\partial t'}dt'. \tag{4.13}$$

A few comments are in order concerning Eq. (4.13). First of all $\rho_{\rm GW}(\omega_m,t)$ denotes the energy density in GWs [see Eq. (4.12)] integrated until a “running” ultraviolet cutoff

$\omega_m(t)\sim 1/t\sim H(t)$. The partial derivatives appearing in Eq. (4.13) simply reflect the fact that we are summing up the energy of GWs reentering the horizon at different moments $t>t_1$. In order to do this we have to exactly slice the horizon in many infinitesimal portions and compute, for each portion, the corresponding increment in the GW energy. In computing the increment in the GW energy with respect to the cutoff we keep the scale factor constant which is the reason for the subscript appearing in Eq. (4.13). We now want to compute the frequency integral of Eq. (4.12). As is well known this integral is divergent in the limit of large frequencies [35]. This is simply a consequence of the fact that the vacuum leads to an infinite energy density which has to be properly subtracted. The spectrum of the vacuum fluctuations can be simply obtained by putting $|C_-(\omega,t_1)|=0$ in Eq. (4.11). This limit corresponds to the absence of amplification and therefore the only fluctuations contributing to the energy density are the ones associated with the vacuum modes with logarithmic energy spectrum proportional to ω^4. In principle, in our case we have an (physical) ultraviolet cutoff in the spectrum provided by the scale where inflation stops. This maximal frequency is $k\simeq \eta_1^{-1}$. We then expect our results to be insensitive to the particular renormalization scheme. Since, however, we want to have [in Eq. (4.13)] the possibility of a cutoff running with time we will briefly examine this issue which has actually been investigated in the past for homogeneous cosmological backgrounds using (at least) two different approaches. In Ref. [34] this problem was tackled using a regularization scheme strongly reminiscent of the Pauli-Villars method. In Ref. [36] the same problem was discussed within the so-called adiabatic regularization scheme (see also Ref. [22]). The two methods were shown to produce equivalent results [36]. The purpose of this paper is not to check which regularization scheme is better to use in curved space. Our approach is more pragmatic: we want to get an estimate not only of the maximal duration of the stiff phase but also of the transition time between the stiff and radiation dominated epochs. We want to know how long it will take for the stiff fluid to turn into radiation. In this spirit we will first of all use the adiabatic regularization scheme (without including higher order subtractions involving double time derivatives of the Hubble parameter). Secondly we will recompute our back reaction effects on the background evolution without including the subtractions and using a (naive) cutoff regularization.

In the adiabatic scheme the regularized energy density reads

$$\overline{\rho}_{\rm GW}(t)=\frac{1}{8\pi^3}\int \omega d^3\omega[F_1(\omega,t)-F_2(\omega,t)], \quad (4.14)$$

where

$$F_1(\omega,t)=\frac{1}{2}\left[\left(\frac{\Omega_\omega(t)}{\Omega_\omega(t_1)}\right)+\left(\frac{\Omega_\omega(t_1)}{\Omega_\omega(t)}\right)\right],$$
$$F_2(\omega,t)=\left[1+\frac{1}{2}\left(\frac{H}{\omega}\right)^2+\frac{1}{8}\frac{\Sigma(t)}{\omega^4}\right]. \quad (4.15)$$

We notice that $F_1(\omega,t)$ is simply the nonregularized contribution and is exactly equal to the one we computed [Eq. (4.11)]. On the other hand, $F_2(\omega,t)$ comes from the adiabatic regularization [36] and contains the divergent contribution which we want to subtract. Notice that in the expansion there are terms with higher derivatives since

$$\Sigma(t)=[\dot{H}^2+2H^4-2\ddot{H}H-4H^2\dot{H}]. \quad (4.16)$$

Now, our underlying theory is the Einstein theory [with only linear curvature terms in the action (3.5)]. Consequently, it is not compatible to include subtractions involving four derivatives. Therefore we will not include them in the subtraction. We can integrate the energy density keeping a running ultraviolet cutoff $\omega_m\sim H(t)\lesssim H_{t_1}$

$$\overline{\rho}_{\rm GW}(t)=\frac{1}{4\pi^2}\int^{\omega_m(t)}\omega^4\left\{\left[\left(\frac{a}{a_1}\right)-\left(\frac{a_1}{a}\right)^2\right]^2-\frac{H^2}{\omega^2}\right\}d\log\omega, \quad (4.17)$$

with the result that

$$\overline{\rho}_{\rm GW}(t)=\frac{\omega_m^4}{16\pi^2}\left\{\left[\frac{a}{a_1}-\frac{a_1}{a}\right]^2-2\left(\frac{H}{\omega_m}\right)^2\right\}. \quad (4.18)$$

We are now ready to include the effect of the produced gravitational waves in the Einstein equations. The evolution of the Hubble parameter will be given (in the conformally flat case) by the following integrodifferential equation:

$$M_P^2H^2-\rho_s(t)=-\int_{t_1}^{t}\left[\frac{a(t')}{a(t)}\right]^4\left[\frac{\partial\overline{\rho}_{\rm GW}(\omega_m(t'),t')}{\partial\omega_m}\right]_a\frac{\partial\omega_m}{\partial t'}dt', \quad (4.19)$$

$$\rho_s(t)=H_1^2M_P^2\left(\frac{a_1}{a}\right)^{3(\gamma+1)},$$

$$\left[\frac{\partial\overline{\rho}_{GW}}{\partial\omega_m}\right]_a=\frac{\omega_m^3}{4\pi^2}\left\{\left[\left(\frac{a}{a_1}\right)-\left(\frac{a_1}{a}\right)\right]^2-\frac{H^2}{\omega_m^2}\right\}. \quad (4.20)$$

In Eq. (4.20) ρ_s is just the energy density of the stiff background which decreases faster that a^{-4} for any $\gamma>1/3$. Notice that the right-hand side of Eq. (4.19) is nothing but the first of the FRW equations reported in Eq. (2.2). In the absence of graviton creation the left-hand side of Eq. (4.19) would just be zero, whereas in the presence of a graviton reentering the horizon at any time after t_1 this second term receives a nonvanishing contribution.

A useful way of rewriting the integral appearing in Eq. (4.19) is by changing the integration variable from t' to $H'\equiv H(t')$, and in this way the total energy of produced gravitons becomes

$$\rho_{\rm tot}(t)=-\frac{1}{4\pi^2}\int_{H_1}^{H}\left[\frac{a(H')}{a(H)}\right]^4\omega_m^3\left\{\left[\frac{a}{a_1}-\frac{a_1}{a}\right]^2-\left(\frac{H'}{\omega_m}\right)^2\right\}dH'. \quad (4.21)$$

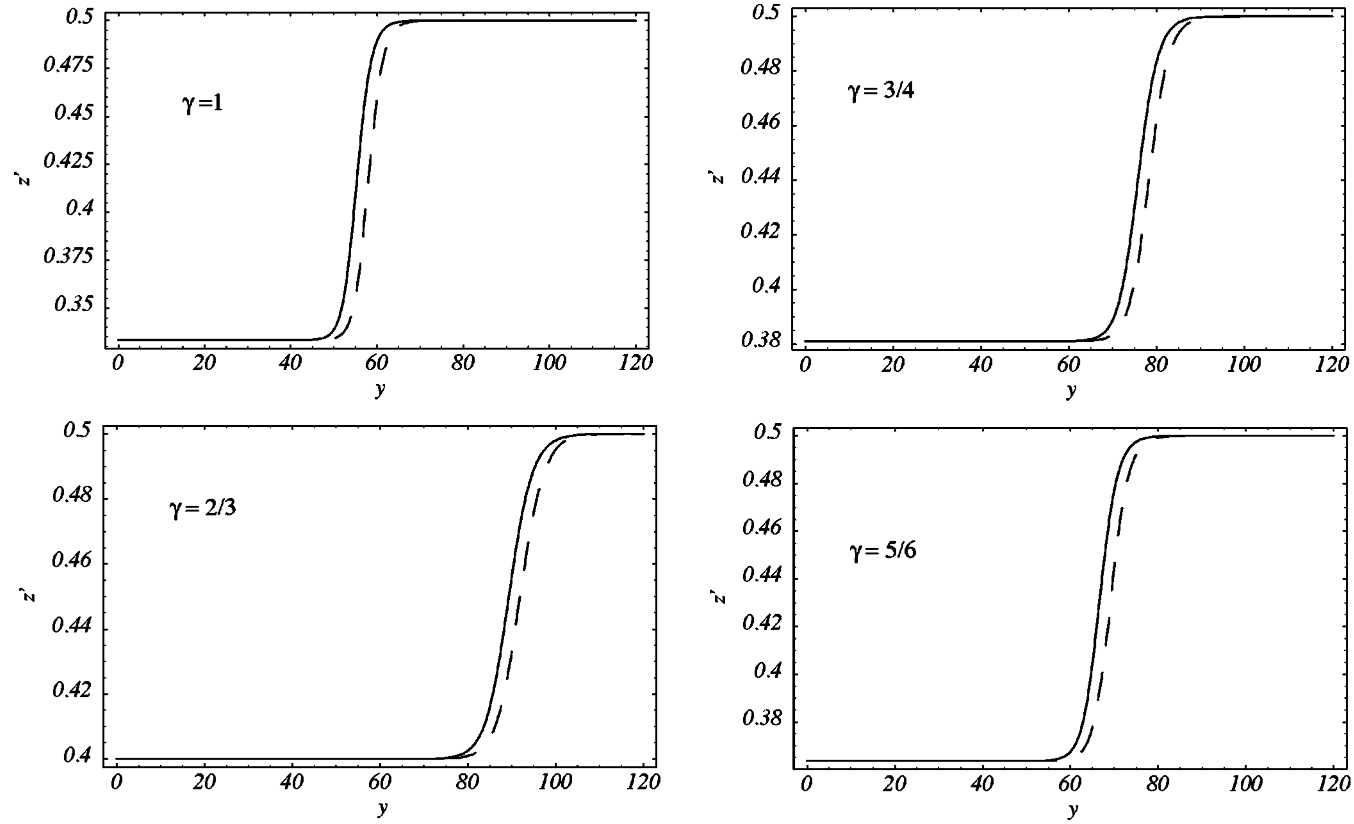

FIG. 2. We plot $z'(y)$ in the case $\epsilon = 10^{-7}$. Recall that $z = \log(a/a_1)$ and $y = \log(t/t_1)$. Therefore, we have that $\alpha(y) = z'(y)$. We plot the solutions for four different values of the initial γ parameter. We integrated Eq. (4.23) from $t = t_1$ (i.e., $y = 0$) up to $y = 120$. Take, for instance, the case $\gamma = 1$ corresponding to $\alpha(t_1) = 1/3$. We see that, thanks to the gravitational wave contribution, the background undergoes a phase transition towards a radiation dominated epoch with $z'(y) \sim \alpha(y) \sim 1/2$. In the full lines the case $q = 0$ is reported (no subtractions of the ultraviolet divergences) whereas in the full lines the case $q = 1$ is illustrated including subtractions.

Notice that $\partial \omega_m / \partial H \sim 1$ since $\omega_m \sim H$ when the given mode crosses the horizon.

It is clear that Eq. (4.20) represents a complicated integrodifferential equation which cannot be solved exactly. One possible way of dealing with this problem is to transform it into an ordinary differential equation by solving the integral using the scale factor of the stiff phase. In other words, the integral in the right-hand side of Eq. (4.19) can be viewed as a perturbation to the solution of the FRW equations (2.2) in the absence of graviton creation. Using this procedure we insert the stiff scale factor into Eq. (4.20) and we compute the effect of the gravitons reentering after t_1 on the background metric. Thus using $[a(t')/a(t_1')] \sim [t'/t_1]^{2/3(\gamma+1)}$, $\omega_m \sim 1/t'$, and $H' = H(t') \sim 2/[3(\gamma+1)t']$ in Eq. (4.21), our integrodifferential equation becomes

$$\left(\frac{dz}{dy}\right)^2 = e^{2y}\left[\frac{4}{9(\gamma+1)^2} e^{-2y} + \epsilon^2 e^{-4x} \Lambda(z)\right]$$

$$\Lambda(z) = \{f_1(\gamma)[e^{-2(1+3\gamma)z} - 1] + f_2(\gamma)[1 - e^{-2(3\gamma+2)z}] + f_3(\gamma)[1 - e^{-6\gamma z}]\}, \tag{4.22}$$

where

$$f_1(\gamma) = \frac{1}{24\pi^2} \frac{2q + 9(1+\gamma)}{(\gamma+1)(3\gamma+1)^2},$$

$$f_2(\gamma) = \frac{1}{16\pi^2} \frac{3(\gamma+1)}{2+3\gamma}, \quad f_3(\gamma) = \frac{1}{16\pi^2}\left(\frac{\gamma+1}{\gamma}\right) \tag{4.23}$$

(recall that $\epsilon = H_1/M_P$ [37]). In Eq. (4.23) there is also the parameter q which needs to be explained. As we said before we regularized the energy density using the adiabatic regularization scheme. Now, if we set $q = 1$ we automatically include in Eq. (4.23) the subtractions coming from the adiabatic regularization. If we set $q = 0$ we practically use for the calculation of the energy density of the gravitational waves the nonregularized energy density. Thus, if $q = 1$, $\overline{\rho}_{GW}$ is used in Eq. (4.20). If $q = 0$, Eq. (4.20) is computed by using ρ_{GW} which does not include the subtractions of the divergent terms. As we discussed previously, setting $q = 1$ or $q = 0$ does not change the numerical solution which we are going to describe. It should actually be borne in mind that we have a physical cutoff provided by the class of models we are discussing and which is set by $k_1 \sim \eta_1^{-1}$. The effect of the subtractions (encoded in the choice of q) is illustrated in Fig. 2 where the solution of Eq. (4.23) is reported as a function of $y = \log t/t_1$. We can see that to include the subtractions affects

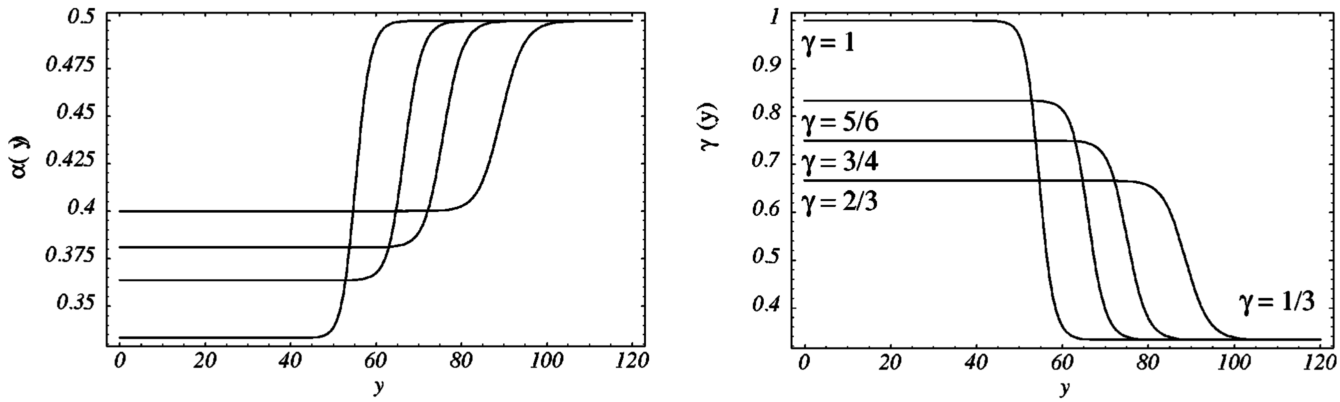

FIG. 3. The time evolution of the equation of state is reported. Recall, for comparison between the two pictures that $\gamma(y) = 2/[3z'(y)] - 1$ and $\alpha(y) = z'(y)$. The four curves do correspond to the four cases already discussed in Fig. 2. We notice that starting with stiff equations of state at $t = t_1$ [i.e., from top to bottom, $\gamma(0) = 1,\ 5/6,\ 3/4,\ 2/3$] we are attracted towards $\gamma = 1/3$ for large y.

the transition regime (when the gravitational radiation starts to dominate the background) but not the asymptotic regime. Notice that from the effective evolution of $\alpha(y)$ we can also derive the effective evolution of $\gamma(y)$. In Fig. 3 the cosmic time evolution of $\gamma(y)$ is illustrated for the four different cases of $\gamma(0)$ discussed in Fig. 2. The back reaction which forces the equation of state to pass from its original stiff value (for $y \to 0$) to 1/3 the typical value of a radiation dominated (perfect) relativistic fluid.

It is also clear that by lowering ϵ the transition to the radiation dominated epoch might be delayed (paying, of course, the price of a fine-tuning). In Fig. 4 we report the solution of Eq. (4.23) for different values of ϵ in the case $\gamma = 1$. As usual we can either see the transition in terms of γ or α. Looking at Figs. 3 and 4 we can also see that the transition to the radiation dominated phase does not occur instantaneously but takes place in a finite amount of time which turns out to be quite substantial (of the order of 20 times e folding).

V. DISCUSSION AND CONCLUSIONS

In this paper we investigated the possibility of stiff epochs occurring immediately after an inflationary phase. We found that if these phases exist they are constrained by the production of GWs. We computed the associated energy spectra of the produced gravitons and we also discussed the associated back reaction effects. Concerning the theoretical implications of GW backgrounds of stiff origin specifically, we can say that the possibility of having blue spectra at high frequencies certainly looks promising. In this class of models blue spectra arise quite naturally. If the maximal inflationary curvature scale is taken to be of the order of $10^{-6} M_P$ we can also see from our results that the amplitude of the GW (logarithmic) energy spectrum will be larger, at high frequencies, than the inflationary prediction obtained in the absence of stiff phases. The back reaction effect will make the Universe dominated by radiation and the corresponding ''graviton reheating'' energy scale in the range $1-10^4$ TeV.

The duration of the stiff phase can be long only if the maximal scale where inflation occurs is fine-tuned to be much smaller than $10^{-6} M_P$. We also found that the transition regime (where the stiff equation of state with $\gamma > 1/3$) is replaced by $\gamma = 1/3$ is quite long. Our analysis certainly has different limitations. We mainly focused our attention on the back reaction effects associated with high-frequency gravitons which behave effectively as radiation [20–22]. The second limitation of our analysis is that we did not discuss the

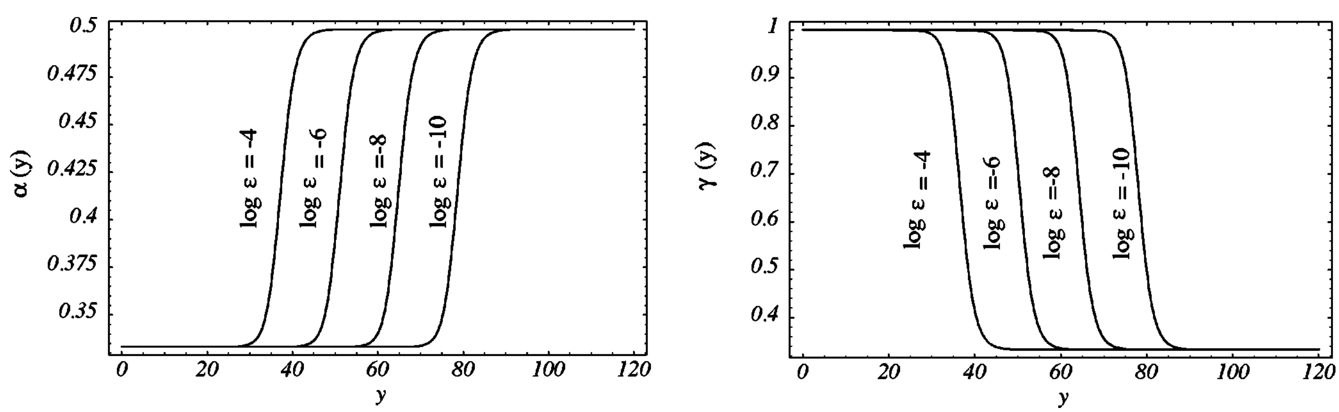

FIG. 4. The solution of Eq. (4.23) is reported for the case $\gamma = 1$. If $\epsilon = 10^{-4}$ (as required by BBN constraints) we have that the equation of state of the fluid driving the expansion starts to deviate significantly from the stiff one already for $y \sim 4 \times 10^9$ corresponding to $H_r \sim 10^{-13} M_P$ (notice that we always use Neperian logarithms).

amplification of the scalar fluctuations. We did not do this for the simple reason that the scalar fluctuations are much more sensitive to the particular dynamical model used in order to implement a stiff phase.

A conservative conclusion of our investigation is that the the duration of a stiff post-inflationary phase is crucially determined by the energy density of the inhomogeneity which was amplified during the inflationary phase and reentered in the stiff phase. In the framework of a particular model the back reaction effects of tensor (and scalar) fluctuations should be analyzed. On one hand, the produced inhomogeneities could offer an original mechanism for reheating the Universe. On the other hand, they could forbid an arbitrary duration of the stiff phase.

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